An Approximate Riemann Solver for Capturing Free-Surface Water Waves

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ABSTRACT
In the instance of two-phase flow, the shock capturing ability of Godunov-type schemes may serve to maintain robustness and accuracy at the interface. Approximate Riemann solvers have relieved the initial drawback of computational expensiveness of Godunov-type schemes. In this paper we present an Osher-type approximate Riemann solver for application in hydrodynamics. Actual computations are left to future research.

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1. Introduction

The advantages of Godunov-type schemes [1] in hydrodynamic flow computations are not as widely appreciated as in gas dynamics applications. Admittedly, the absence of supersonic speeds and hence shock waves in incompressible flow (the prevailing fluid model in hydrodynamics) reduces the necessity of advanced shock capturing schemes. Nevertheless, many reasons remain to apply Godunov-type schemes in hydrodynamics: Firstly, these schemes have favourable robustness properties due to the inherent upwind treatment of the flow. Secondly, they feature a consistent treatment of boundary conditions. Thirdly, (higher-order accurate) Godunov-type schemes display low dissipative errors, which is imperative for an accurate resolution of boundary layers in viscous flow. Finally, the implementation of these schemes in conjunction with higher-order limited interpolation methods, to maintain accuracy and prevent oscillations in regions where large gradients occur (see, e.g., [2, 3]), is relatively straightforward.

In addition, Godunov-type schemes can be particularly useful in hydrodynamics in case of two-phase flows, e.g., flows suffering cavitation and free surface flows. In these situations, an interface exists between the primary phase (water) and the secondary phase (air, damp, etc.) and fluid properties may vary discontinuously across the interface. In our opinion, the ability of Godunov-type schemes to capture discontinuities is then very useful to maintain robustness and accuracy at the interface. Examples of such interface capturing can be found in, e.g., [4, 5, 6].

A disadvantage of the method originally proposed by Godunov is that it requires the solution of an associated Riemann problem with each flux evaluation. In practice, many such evaluations are performed during an actual computation. Consequently, the method is notorious for its high computational costs. To relieve this problem, several approaches have been suggested to reduce the computational costs of the flux evaluations involved, by approximating the Riemann solution. Examples of such approximate Riemann solvers are the flux difference splitting schemes (such as Roe's [7] and Osher's [8]).

In the paper we present an Osher-type flux-difference splitting scheme for the Euler equations describing a two-phase fluid flow. As a preliminary, we give an outline of Osher's approximate Riemann solver. Analysis shows that the scheme suffers loss of accuracy in the presence of centered shock waves and therefore a modified scheme is proposed. Finally, we present the specifics for the aforementioned hydrodynamic application. Actual computations are deferred to future research.

2. Approximate Riemann Solution

Definition 2.1 *Let* $\mathbf{q} \in \mathbb{R}^n = (q_1, \ldots, q_n)^T$, $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ *and* $\mathbf{f} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. *Consider the Cauchy problem*

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial x} = 0, \qquad \forall x \in \mathbb{R}, t \in \mathbb{R}^+, \tag{2.1a}$$

subject to the initial condition

$$\mathbf{q}(x,0) = \begin{cases} \mathbf{q}_L, & \text{if } x < 0, \\ \mathbf{q}_R, & \text{if } x > 0, \end{cases} \tag{2.1b}$$

with $\mathbf{q}_L$ *and* $\mathbf{q}_R$ *constant. The initial value problem (2.1a) and the initial condition (2.1b) define the Riemann problem.*

In the previous section we established that the solution to the Riemann problem can generally be written in similarity form $\mathbf{h}(x/t)$. Denoting by $\mathbf{h}(x/t; \mathbf{q}_L, \mathbf{q}_R)$ the similarity solution for given $\mathbf{q}_L$ and $\mathbf{q}_R$, we find $\mathbf{f}(\mathbf{h}(0; \mathbf{q}_L, \mathbf{q}_R))$ to be the corresponding centered flux, $\mathbf{f}(\mathbf{q}_L, \mathbf{q}_R)$. This flux is of particular importance in computational applications: following Godunov's approach, it can be interpreted as the flux between two adjacent cells in the discretised domain. Unfortunately, solving the Riemann problem exactly is computationally expensive and it is therefore necessary to revert to approximate solution techniques.

In this section, we investigate Osher's approximate Riemann solver and a modified Osher-type scheme. We will first present a general outline of the Osher scheme. Subsequently, the approximate Riemann solution employed in Osher's scheme is examined and the computed flux approximation is compared to the exact solution. Finally, we shall propose the modified scheme, based on the preceding analysis.

2.1 Osher's scheme

In the scheme developed by Osher [8, 9], the centered flux $\mathbf{f}(\mathbf{q}_L, \mathbf{q}_R) = \mathbf{f}(\mathbf{h}(0; \mathbf{q}_L, \mathbf{q}_R))$ is approximated by:

$$\tilde{\mathbf{f}}(\mathbf{q}_L, \mathbf{q}_R) = \frac{1}{2}\mathbf{f}(\mathbf{q}_L) + \frac{1}{2}\mathbf{f}(\mathbf{q}_R) - \frac{1}{2}\int_{\mathbf{q}_L}^{\mathbf{q}_R} |\mathbf{A}(\mathbf{w})| \cdot \mathrm{d}\mathbf{w}, \tag{2.2}$$

with the absolute value of the Jacobian matrix $\mathbf{A}(\mathbf{q})$ defined by $|\mathbf{A}(\mathbf{q})| \equiv \mathbf{R}(\mathbf{q}) \cdot |\mathbf{\Lambda}(\mathbf{q})| \cdot \mathbf{R}(\mathbf{q})^{-1}$. Here, $|\mathbf{\Lambda}(\mathbf{q})| = \mathrm{diag}(|\lambda_1(\mathbf{q})|, \ldots, |\lambda_n(\mathbf{q})|)$. Clearly, the integral term represents the upwind contribution to the centered flux approximation.

The integral in (2.2) is evaluated along a path $\Gamma = \{\mathbf{q}(s) : 0 \leq s \leq 1\} \subset \mathbb{R}^n$ in state space, satisfying $\mathbf{q}(0) = \tilde{\mathbf{q}}_0$ and $\mathbf{q}(1) = \tilde{\mathbf{q}}_1$, with $\tilde{\mathbf{q}}_0 = \mathbf{q}_L$ and $\tilde{\mathbf{q}}_1 = \mathbf{q}_R$ or vice versa. This path is composed of sub-paths Γ_l, $l = 1, 2, \ldots, n$, where each of the sub-paths connects two adjacent states $\tilde{\mathbf{q}}_{(l-1)/n}$ and $\tilde{\mathbf{q}}_{l/n}$. Moreover, Γ_l is tangential to an eigenvector $\mathbf{r}_{k(l)}$, where $k : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a bijective mapping. It should be appreciated here that Γ_l is thus a section of the $k(l)$-path through $\tilde{\mathbf{q}}_{(l-1)/n}$, connecting $\tilde{\mathbf{q}}_{(l-1)/n}$ and $\tilde{\mathbf{q}}_{l/n}$. Usual choices for the ordering of the sub-paths are the O-variant $k(l) = n - l$ and the P-variant $k(l) = l$.

The integral term in (2.2) can be rewritten as a summation of the contributions of the integral over each of the sub-paths:

$$\int_{\mathbf{q}_L}^{\mathbf{q}_R} |\mathbf{A}(\mathbf{w})| \cdot \mathrm{d}\mathbf{w} = \sum_{l=1}^{n} \int_{\Gamma_l} |\mathbf{A}(\mathbf{w}(\xi))| \cdot \mathbf{r}_{k(l)}(\mathbf{w}(\xi)) \mathrm{d}\xi = \sum_{l=1}^{n} \int_{\Gamma_l} \mathrm{sign}(\lambda_{k(l)}(\mathbf{w})) \mathbf{A}(\mathbf{w}) \cdot \mathrm{d}\mathbf{w}. \tag{2.3}$$

Obviously, if $\lambda_{k(l)}$ does not change sign along Γ_l, then the sub-integral can be evaluated to $[\mathbf{f}(\tilde{\mathbf{q}}_{l/n}) - \mathbf{f}(\tilde{\mathbf{q}}_{(l-1)/n})]\,\mathrm{sign}(\lambda_{k(l)})$. Then, if $\lambda_{k(l)} = \lambda_{k(l+1)} = \ldots = \lambda_{k(l+\mu)}$ is a linearly degenerate eigenvalue, the sum in (2.3) concatenates and we simply obtain

$$\sum_{i=0}^{\mu} \int_{\Gamma_{l+i}} |\mathbf{A}(\mathbf{w})| \cdot \mathrm{d}\mathbf{w} = \mathrm{sign}(\lambda_{k(l)}(\mathbf{q}_{l/n}))[\mathbf{f}(\mathbf{q}_{l/n}) - \mathbf{f}(\mathbf{q}_{(l+\mu)/n})]. \tag{2.4}$$

Hence, the intermediate stages $\tilde{\mathbf{q}}_{(l+i)/n}$, $i = 1, 2, \ldots, \mu-1$ are of no consequence and may be eliminated from the composed path Γ.

As a result of the choice of the sub-paths Γ_l, the intermediate $\tilde{\mathbf{q}}_{l/n}$, $l = 1, 2, \ldots, n-1$ can be conveniently determined by means of the Riemann invariants: Because the sub-path $\Gamma_l \subset \mathcal{R}_{k(l)}(\tilde{\mathbf{q}}_{(l-1)/n})$,

$$\psi^m_{k(l)}(\tilde{\mathbf{q}}_{(l-1)/n}) = \psi^m_{k(l)}(\tilde{\mathbf{q}}_{l/n}), \qquad l, m = 1, 2, \ldots, n, \quad m \neq k(l). \tag{2.5}$$

If it is assumed that the k-Riemann invariants in (2.5) have linearly independent gradients, then by the implicit function theorem, (2.5) constitutes a solvable system of equations from which the $\tilde{\mathbf{q}}_{l/n}$, $l = 1, 2, \ldots, n$ can be extracted. In many practical cases the intermediate stages can then be solved explicitly from (2.5). Once the intermediate states $\tilde{\mathbf{q}}_{l/n}$ have been obtained, the flux approximation $\tilde{\mathbf{f}}(\mathbf{q}_L, \mathbf{q}_R)$ can be computed using (2.2), (2.3).

2.2 Accuracy

The flux computed by means of the Osher scheme, $\tilde{\mathbf{f}}(\mathbf{q}_L, \mathbf{q}_R)$, relies on an approximate solution of the Riemann problem. Because the approximation can again be written in similarity form, it is useful to introduce the notation $\tilde{\mathbf{f}}(\mathbf{q}_L, \mathbf{q}_R) = \mathbf{f}(\tilde{\mathbf{h}}(0; \mathbf{q}_L, \mathbf{q}_R))$, where $\tilde{\mathbf{h}}(x/t; \mathbf{q}_L, \mathbf{q}_R)$ stands for the approximate similarity solution. In this section we investigate the accuracy of the approximate similarity solution and of the corresponding centered flux approximation.

To evaluate the accuracy of the approximate solution, we examine the inherent representation of simple waves, contact discontinuities and shock waves. In section 2.1 it was emphasized that the sub-paths, Γ_l, in Osher's scheme are actually sections of $k(l)$-paths. It follows that the intermediate states $\tilde{\mathbf{q}}_{l/n}$, $l = 0, \ldots, n$, in the approximate solution are connected by simple waves only. Clearly, this representation is correct for simple waves and contact discontinuities. However, shock waves in the actual solution are then replaced by so-called *overturned simple waves*, see [11]. We will now show that this representation is accurate for weak shocks. From [10] we adopt:

Lemma 2.1 *Suppose $\mathbf{q}_L$ and $\mathbf{q}_R$ are connected by a weak k-shock with shock strength ϵ, i.e., $\mathbf{q}_R \in \mathcal{S}_k(\mathbf{q}_L)$ and $\lambda_k(\mathbf{q}_L) = \lambda_k(\mathbf{q}_R) + \epsilon$, with ϵ a small positive number. Then the change in a k-Riemann invariant across the k-shock is of order $\mathcal{O}(\epsilon^3)$.*

Proof: *Proof is omitted here, but can be found in [10, pages 326–333].* □

Then, we obtain:

Theorem 2.1 *Suppose $\mathbf{q}_R \in \mathcal{S}_k(\mathbf{q}_L)$ and $\lambda_k(\mathbf{q}_L) = \lambda_k(\mathbf{q}_R) + \epsilon$. Then a $\tilde{\mathbf{q}}_R \in \mathcal{R}_k(\mathbf{q}_L)$ exists such that $\lambda_k(\tilde{\mathbf{q}}_R) = \lambda_k(\mathbf{q}_R)$ and $|\tilde{\mathbf{q}}_R - \mathbf{q}_R|$ is of order $\mathcal{O}(\epsilon^3)$.*

Proof: *By definition, $\psi^m_k(\mathbf{q}_L) = \psi^m_k(\tilde{\mathbf{q}}_R)$, $k = 1, 2, \ldots, n$, $k \neq m$. Then, by lemma 2.1,*

$$\psi^m_k(\tilde{\mathbf{q}}_R) = \psi^m_k(\mathbf{q}_R) + \mathcal{O}(\epsilon^3). \tag{2.6}$$

System (2.6) can be augmented with $\lambda_k(\tilde{\mathbf{q}}_R) = \lambda_k(\mathbf{q}_R)$ to obtain n equations for $\tilde{\mathbf{q}}_R$. Because $\mathrm{rank}(\partial_{\mathbf{q}}\psi^1_k, \ldots, \partial_{\mathbf{q}}\psi^n_k) = n - 1$ and $\partial_{\mathbf{q}}\lambda_k \in (\partial_{\mathbf{q}}\psi^1_k, \ldots, \partial_{\mathbf{q}}\psi^n_k)^{\perp}$, $\det(\partial_{\mathbf{q}}\psi^1_k, \ldots, \partial_{\mathbf{q}}\psi^n_k, \partial_{\mathbf{q}}\lambda_k) \neq 0$. The result now simply follows by a Taylor expansion around $\mathbf{q}_R$ of the terms in $\tilde{\mathbf{q}}_R$ of the augmented system. □

From Theorem 2.1 it may be inferred that the intermediate states obtained by a rarefaction-waves-only approximation are $\mathcal{O}(\epsilon_{\max}^3)$ accurate, with

$$\epsilon_{\max} = \max_{l=1\ldots n} (\lambda_l(\mathbf{q}_{(l-1)/n}) - \lambda_l(\mathbf{q}_{l/n}), 0) \tag{2.7}$$

the strength of the strongest shock.

Although the computed intermediate states are accurate even in the presence of (weak) shocks, the flux approximation $\tilde{\mathbf{f}}(\mathbf{q}_L, \mathbf{q}_R)$ is not necessarily so. By (2.3), if $\tilde{\mathbf{q}}_R \in \mathcal{R}_k(\mathbf{q}_L)$ and $\lambda_k(\mathbf{q}_L) > 0 > \lambda_k(\tilde{\mathbf{q}}_R)$,

$$\tilde{\mathbf{f}}(\mathbf{q}_L, \tilde{\mathbf{q}}_R) = \mathbf{f}(\mathbf{q}_L) + \mathbf{f}(\tilde{\mathbf{q}}_R) - \mathbf{f}(\mathbf{q}^*), \tag{2.8}$$

with $\mathbf{q}^* \in \mathcal{R}_k(\mathbf{q}_L)$ such that $\lambda_k(\mathbf{q}^*) = 0$. In contrast, ignoring terms $\mathcal{O}(\epsilon^3)$, one finds that the actual flux corresponding to the *k-shock* is $\mathbf{f}(\mathbf{q}_L)$ if $s(\mathbf{q}_R; \mathbf{q}_L) > 0$ and $\mathbf{f}(\mathbf{q}_R)$ if $s(\mathbf{q}_R; \mathbf{q}_L) < 0$. Consequently, the error in the approximate flux in the instance of a centered shock with strength ϵ may be of $\mathcal{O}(\epsilon)$.

2.3 Modified Scheme

In view of the above, a modification of the scheme is advocated. The rarefaction-waves-only approximation of the similarity solution is maintained. However, the centered flux approximation is obtained differently, to avoid loss of accuracy due to centered shock waves.

We propose to extract the intermediate states in the approximate solution to the Riemann problem from

$$\psi_l^m(\tilde{\mathbf{q}}_{(l-1)/n}) = \psi_l^m(\tilde{\mathbf{q}}_{l/n}), \qquad l, m = 1, 2, \ldots, n, \quad m \neq l, \tag{2.9}$$

with $\tilde{\mathbf{q}}_0 = \mathbf{q}_L$ and $\tilde{\mathbf{q}}_1 = \mathbf{q}_R$. This is in fact equivalent to (2.5) with a presumed P-variant ordering of the sub-paths. Next, approximate contact speeds $\tilde{\sigma}_l^{\pm}$ are obtained:

$$\tilde{\sigma}_l^{\pm} = \begin{cases} \lambda_{l+(1\pm1)/2}(\tilde{\mathbf{q}}_{l/n}) & \text{if} \quad \pm\lambda_{l+(1\pm1)/2}(\tilde{\mathbf{q}}_{l/n}) < \pm\lambda_{l+(1\pm1)/2}(\tilde{\mathbf{q}}_{(l\pm1)/n}), \\ \tilde{s}_{l+(1\pm1)/2} & \text{otherwise,} \end{cases} \tag{2.10a}$$

with

$$\tilde{s}_{l+(1\pm1)/2} = \frac{1}{2}\lambda_{l+(1\pm1)/2}(\tilde{\mathbf{q}}_{l/n}) + \frac{1}{2}\lambda_{l+(1\pm1)/2}(\tilde{\mathbf{q}}_{(l\pm1)/n}). \tag{2.10b}$$

Estimate (2.10b) of the shock speed is justified by the following theorem, taken from [10]:

Theorem 2.2 *Suppose* $\mathbf{q}_R \in \mathcal{S}_k(\mathbf{q}_L)$ *and* $\lambda_k(\mathbf{q}_L) = \lambda_k(\mathbf{q}_R) + \epsilon$, $\epsilon > 0$. *Then the speed of the k-shock connecting* $\mathbf{q}_L$ *and* $\mathbf{q}_R$ *satisfies* $s(\mathbf{q}_L; \mathbf{q}_R) = \frac{1}{2}\lambda_k(\mathbf{q}_L) + \frac{1}{2}\lambda_k(\mathbf{q}_R) + \mathcal{O}(\epsilon^2)$.

Proof: *Proof can be found in [10, pages 326–333].* □

Once the intermediate states and contact speeds have been established, the approximate Riemann solution can be constructed. However, considering that our purpose is to compute an approximation to the centered flux, we only need to obtain the central part of the approximate solution:

$$\tilde{\mathbf{h}}(0; \mathbf{q}_L, \mathbf{q}_R) = \begin{cases} \tilde{\mathbf{q}}_0, & \text{if} \quad \tilde{\sigma}_0^+ > 0, \\ \tilde{\mathbf{q}}_{l/n}, & \text{if} \quad \tilde{\sigma}_l^- < 0 < \tilde{\sigma}_l^+, \quad l \in \{1, \ldots, n-1\}, \\ \tilde{\mathbf{q}}^*, & \text{if} \quad \tilde{\sigma}_{l-1}^+ < 0 < \tilde{\sigma}_l^-, \quad l \in \{1, \ldots, n-1\}, \\ \tilde{\mathbf{q}}_1, & \text{if} \quad \sigma_n^- < 0, \end{cases} \tag{2.11}$$

with $\tilde{\mathbf{q}}^* \in \mathcal{R}_l(\tilde{\mathbf{q}}_{(l-1)/n})$ such that $\lambda_l(\tilde{\mathbf{q}}^*) = 0$ in case of a centered rarefaction wave. The centered flux approximation is now simply $\tilde{\mathbf{f}}(\mathbf{q}_L, \mathbf{q}_R) = \mathbf{f}(\tilde{\mathbf{h}}(0; \mathbf{q}_L, \mathbf{q}_R))$.

3. Applications in Hydrodynamics

In the previous section we presented a flux-difference splitting scheme that gives an accurate approximation of the centered flux in the Riemann problem, even in the presence of (weak) centered shock waves. A prerequisite for the flux evaluation is the derivation of the intermediate states $\tilde{\mathbf{q}}_{l/n}$, $l = 1, \ldots, n$. Once these states have been obtained, the flux calculation proceeds via straightforward operations.

In this section we derive the intermediate states for the Euler equations in the case of an immiscible, compressible two-phase flow model, which uses a level-set function to distinguish the two fluids from each other.

3.1 Two-Phase Flow

The two phases are supposed to be separated by a moving interface, which is described by the time-dependent set $\mathcal{I}(t) = \{\mathbf{x} \in \mathbb{R}^3 \mid \theta(\mathbf{x}, t) = 0\}$. Furthermore, we assume the level-set function $\theta(\mathbf{x}, t)$ to be negative in one phase and positive in the other. As a result of the immiscibility of the phases, the following kinematic condition applies:

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \vec{\nabla} \theta = 0, \tag{3.1}$$

where $\mathbf{u} \in \mathbb{R}^3$ again denotes the fluid velocity. Employing the continuity equation for compressible fluids, we can restate kinematic condition (3.1) in conservation form:

$$\frac{\partial \rho\theta}{\partial t} + \vec{\nabla} \cdot \rho\theta\mathbf{u} = \rho \left(\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \vec{\nabla} \theta \right) + \theta \left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho\mathbf{u} \right). \tag{3.2}$$

The first term in parentheses vanishes due to (3.1), the second due to continuity. Hence, $\rho\theta$ is a conserved quantity. Suppose that throughout the entire fluid volume the pressure is related to the density via an equation of state of the form $p = p(\theta, \rho)$. Then, again using u, v, w to designate the velocity components relative to a Cartesian coordinate system and ignoring spatial derivatives in y and z direction, we retrieve (2.1a), with $\mathbf{q} = (\rho u, \rho v, \rho w, \rho\theta, \rho)^T$ and

$$\mathbf{f}(\mathbf{q}) = \left(q_1^2/q_5 + p(q_4/q_5, q_5), q_1 q_2/q_5, q_1 q_3/q_5, q_1 q_4/q_5, q_1 \right)^T. \tag{3.3}$$

Equations (2.1a), (3.3) constitute the Euler equations for an immiscible, compressible two-phase flow.

Our first objective now is to derive Riemann invariants for (2.1a), (3.3). We define $c_1 = c_1(\theta, \rho) = \sqrt{\partial p/\partial \theta}$ and $c_2 = c_2(\theta, \rho) = \sqrt{\partial p/\partial \rho}$. Then, the Jacobian of (3.3) reads:

$$\mathbf{A}(\mathbf{q}) = \frac{\partial \mathbf{f}}{\partial \mathbf{q}} = \begin{pmatrix} 2q_1/q_5 & 0 & 0 & c_1^2/q_5 & -q_1^2/q_5^2 - c_1^2 q_4/q_5^2 + c_2^2 \\ q_2/q_5 & q_1/q_5 & 0 & 0 & -q_2 q_1/q_5^2 \\ q_3/q_5 & 0 & q_1/q_5 & 0 & -q_3 q_1/q_5^2 \\ q_4/q_5 & 0 & 0 & q_1/q_5 & -q_4 q_1/q_5^2 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.4}$$

The eigenvalues and eigenvectors of $\mathbf{A}(\mathbf{q})$ are

$$\lambda_1 = q_1/q_5 - c_2, \quad \lambda_{2,3,4} = q_1/q_5, \quad \lambda_5 = q_1/q_5 + c_2, \tag{3.5}$$

and

$$\begin{aligned} \mathbf{r}_1 &= (q_1/q_5 - c_2, q_2/q_5, q_3/q_5, q_4/q_5, 1)^T, \\ \mathbf{r}_2 &= (0, 1, 0, 0, 0)^T, \\ \mathbf{r}_3 &= (0, 0, 1, 0, 0)^T, \\ \mathbf{r}_4 &= \left(q_1 c_1^2, 0, 0, -c_2^2 q_5^2 + c_1^2 q_4, q_5 c_1^2 \right)^T, \\ \mathbf{r}_5 &= (q_1/q_5 + c_2, q_2/q_5, q_3/q_5, q_4/q_5, 1)^T. \end{aligned} \tag{3.6}$$

The eigenvalue λ_k and the eigenvector $\mathbf{r}_k$ are genuinely nonlinear for $k = 1, 5$ and linearly degenerate for $k = 2, 3, 4$. Riemann invariants can now be obtained by solving

$$\partial_{\mathbf{q}}\psi_k(\mathbf{q}) \cdot \mathbf{r}_k(\mathbf{q}) = 0, \qquad \forall \mathbf{q} \in \mathbb{R}^n. \tag{3.7}$$

The solution is:

$$\begin{array}{llllllllll} \psi_1^2 = q_1/q_5 + \phi, & \psi_1^3 = q_2/q_5, & \psi_1^4 = q_3/q_5, & \psi_1^5 = q_4/q_5, \\ \psi_2^1 = q_1/q_5, & \psi_2^3 = q_3, & \psi_2^4 = q_4, & \psi_2^5 = p, \\ \psi_3^1 = q_1/q_5, & \psi_3^2 = q_2, & \psi_3^4 = q_4, & \psi_3^5 = p, \\ \psi_4^1 = q_1/q_5, & \psi_4^2 = q_2, & \psi_4^3 = q_3, & \psi_4^5 = p, \\ \psi_5^1 = q_1/q_5 - \phi, & \psi_5^2 = q_2/q_5, & \psi_5^3 = q_3/q_5, & \psi_5^4 = q_4/q_5, \end{array} \tag{3.8a}$$

with $p = p(\theta, \rho)$ and $\phi = \phi(\theta, \rho)$ defined by

$$\phi(\theta, \rho) = \int_0^\rho \frac{c_2(\theta, \eta)}{\eta} \mathrm{d}\eta. \tag{3.8b}$$

Observe that θ is a k-Riemann invariant for $k \in \{1, 5\}$. Hence, it may be inferred that the phase transition is a contact discontinuity. Moreover, because both u and p are k-Riemann invariants for $k \in \{2, 3, 4\}$, the pressure and the normal velocity component are continuous across the interface.

The intermediate states can now be obtained from (2.9), (3.8). Because the linearly degenerate eigenvalue q_1/q_5 has algebraic multiplicity 3, only two intermediate states have to be distinguished. Trivially,

$$\begin{pmatrix} \tilde{v}_{1/3} \\ \tilde{w}_{1/3} \\ \tilde{\theta}_{1/3} \end{pmatrix} = \begin{pmatrix} v_0 \\ w_0 \\ \theta_0 \end{pmatrix}, \qquad \begin{pmatrix} \tilde{v}_{2/3} \\ \tilde{w}_{2/3} \\ \tilde{\theta}_{2/3} \end{pmatrix} = \begin{pmatrix} v_1 \\ w_1 \\ \theta_1 \end{pmatrix}, \tag{3.9}$$

and $\tilde{u}_{1/3} = \tilde{u}_{2/3} \equiv \tilde{u}_{1/2}$. Then, $\tilde{\rho}_{1/3}$ and $\tilde{\rho}_{2/3}$ are determined by

$$\begin{array}{rcl} \phi(\theta_0, \tilde{\rho}_{1/3}) + \phi(\theta_1, \tilde{\rho}_{2/3}) & = & u_0 - u_1 + \phi(\theta_0, \rho_0) + \phi(\theta_1, \rho_1), \\ p(\theta_0, \tilde{\rho}_{1/3}) & = & p(\theta_1, \tilde{\rho}_{2/3}). \end{array} \tag{3.10}$$

We refrain from a further reduction of these expressions and suffice by stating that once the intermediate densities have been obtained, $\tilde{u}_{1/2}$ follows by straightforward computation.

For free-surface water-flow computations, as specific equation of state, we apply

$$\rho = \rho(p, \theta) = \alpha(\theta)\rho_{\text{water}}(p) + (1 - \alpha(\theta))\rho_{\text{air}}(p). \tag{3.11}$$

Here, α is the volume-of-water fraction, which can be accurately computed from the level-set function θ, $\rho_{\text{water}}(p)$ is given by Tait's equation of state and $\rho_{\text{air}}(p)$ by the homentropic perfect gas law.

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